

Weak consistency of modified versions of Bayesian Information Criterion in a sparse linear regression with non-normal error term

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Abstract

We consider a sparse linear regression model, when the number of available predictors p is much bigger than the sample size n and the number of non-zero coefficients p_0 is small. To choose the regression model in this situation, we cannot use classical model selection criteria. In recent years, special methods have been proposed to deal with this type of problem, for example modified versions of Bayesian Information Criterion, like mBIC or mBIC2. It was shown that these criteria are consistent under the assumption that both n and p as well as p_0 tend to infinity and the error term is normally distributed [6]. In this article we prove the consistency of mBIC and mBIC2 with the assumption that the error term is a subgaussian random variable.

Consider the following linear model:

$$y_i = \sum_{j=1}^p \beta_j x_{ij} + \varepsilon_i, \quad i = 1, \dots, n, \quad (1)$$

where ε_i are independent subgaussian random variables with the mean 0 and the variance 1, that is for all $t \in \mathbb{R}$ we have $E e^{t\varepsilon_i} \leq e^{t^2/2}$.

We denote by s a subset of $\{1, \dots, p\}$ (a model), and by $|s|$ the size of s . The true model is s_0 and $|s_0| = p_0$. Our goal is to identify significant predictors. For this purpose, we can use the model selection criteria specially designed for the considered situation, like mBIC [1] or mBIC2 [2, 3, 7], in which we choose the model minimizing the following formulas:

$$\begin{aligned} \text{mBIC}(s) &= n \ln(\text{RSS}(s)) + |s| \ln n + 2|s| \ln p, \\ \text{mBIC2}(s) &= n \ln(\text{RSS}(s)) + |s| \ln n + 2|s| \ln p - 2 \ln |s|!, \end{aligned}$$

where $\text{RSS}(s)$ is the residual sum of squares. We want to show that, with appropriate assumptions, both criteria are consistent. First, we have to introduce additional notations. By $X(s)$ we denote a matrix composed of columns of X with indices s . Next, let $H(s)$ be the projection matrix onto the space spanned by the columns of $X(s)$, that is $H(s) = X(s)[X(s)^T X(s)]^{-1} X(s)^T$. Finally, let $\Delta(s) = \mu^T [I - H(s)] \mu$, where $\mu = X(s_0) \beta(s_0)$.

Before we formulate theorems, we prove the following lemma.

Lemma 1. Consider the set \mathcal{A} consisting of symmetric idempotent matrices. Assume that their maximum rank is r and there are no more than $\binom{p}{j}$ matrices of rank j , $p > r$. Let $m_j = 2j(\ln p + \sqrt{2 \ln p})$. Then, if r tends to infinity, for vectors u composed of subgaussian random variables with variance 1, the following inequalities hold:

$$\max_{A_j} \{u^T A_j u\} \leq m_j + o_P(1),$$

where maximum is taken over all matrices with rank j from \mathcal{A} , a random variable $o_P(1)$ converges in probability to 0 and does not depend on j , and

$$\max_A \{u^T A u\} \leq m_r + o_P(1),$$

where maximum is taken over all matrices from \mathcal{A} .

Proof. In [4] we have the following inequality for a quadratic form $\|Au\|^2$, for all $t > 0$:

$$\mathbb{P} \left(\|Au\|^2 > \text{Tr}(A^T A) + 2\sqrt{\text{Tr}(A^T A)^2 t} + 2\|A^T A\|t \right) \leq \exp(-t), \quad (2)$$

where $\|A^T A\|$ is the spectral norm of the matrix $A^T A$. In the case of matrices from \mathcal{A} , we have $A_j^T A_j = A_j$, $\text{Tr} A_j = j$ and $\|A_j^T A_j\| = 1$ (the largest eigenvalue of A_j). Thus, the inequality (2) can be represented in the following way:

$$\mathbb{P} \left(u^T A_j u > j + 2\sqrt{j t} + 2t \right) \leq \exp(-t).$$

Let $m = j + 2\sqrt{j t} + 2t$. The above inequality can be written as

$$\mathbb{P} \left(u^T A_j u > m \right) \leq \exp \left(-\frac{1}{2}m + \frac{1}{2}\sqrt{2mj - j^2} \right),$$

if $j < m$, but we will employ simpler version:

$$\mathbb{P} \left(u^T A_j u > m \right) \leq \exp \left(-\frac{1}{2}m + \frac{1}{2}\sqrt{2mj} \right). \quad (3)$$

Using the inequality (3) for $m = m_j$ (of course $j < m_j$), we get

$$\begin{aligned} \mathbb{P} \left(\max_{j < r} \{ \max_{A_j} \{ u^T A_j u \} \} > m_j \right) &\leq \sum_{j=1}^r \binom{p}{j} \mathbb{P} \left(\varepsilon^T A_j \varepsilon > m_j \right) \\ &\leq \sum_{j=1}^r p^j \exp \left(-j \ln p - j\sqrt{2 \ln p} + j\sqrt{\ln p + \sqrt{2 \ln p}} \right) \\ &= \sum_{j=1}^r \exp \left(-j\sqrt{2 \ln p} + j\sqrt{\ln p + \sqrt{2 \ln p}} \right) \leq \sum_{j=1}^r \exp \left(-\frac{j}{3}\sqrt{\ln p} \right). \end{aligned}$$

We start summation from 1 because the term for $j = 0$ in the first inequality equals 0. The last inequality is true if p is big enough. We obtain a geometric series $\sum_{j=1}^r (\exp(-\frac{1}{3}\sqrt{\ln p}))^j$, of which sum is

$$\exp\left(-\frac{1}{3}\sqrt{\ln p}\right) \frac{1 - \exp(-\frac{r}{3}\sqrt{\ln p})}{1 - \exp(-\frac{1}{3}\sqrt{\ln p})}$$

and tends to 0. Therefore

$$\max_{A_j} \{u^T A_j u\} \leq m_j + o_P(1),$$

and because the maximum was taken over all $j < r$, the term $o_P(1)$ does not depend on j .

The second inequality in the thesis of the lemma follows from

$$\begin{aligned} P\left(\max_A \{u^T A u\} > m_r\right) &= P\left(\max_{j < r} \{\max_{A_j} \{u^T A_j u\}\} > m_r\right) \\ &\leq P\left(\max_{j < r} \{\max_{A_j} \{u^T A_j u\}\} > m_j\right) \rightarrow 0. \end{aligned}$$

□

Let us now turn to the main theorems, starting with the mBIC.

Theorem 1. Assume the model (1), $p > n$, $p = o(n^{\ln n})$, $p_0 = O\left(\frac{n}{\ln^2 n}\right)$ and

$$\lim_{n \rightarrow \infty} \min \left\{ \frac{\Delta(s)}{p_0 \ln p} : s_0 \not\subset s, v(s) \leq k p_0 \right\} = \infty, \quad (4)$$

where k is any constant > 0 . Then, for n tends to infinity,

$$P\left(\min_{\substack{s: v(s) \leq k p_0 \\ s \neq s_0}} \text{mBIC}(s) > \text{mBIC}(s_0)\right) \rightarrow 1.$$

Proof. In the proof we use similar techniques as in [5]. At the beginning, assume that a set s does not contain the true model, that is $s_0 \not\subset s$. Let us see when $\text{mBIC}(s) > \text{mBIC}(s_0)$:

$$\begin{aligned} &\text{mBIC}(s) - \text{mBIC}(s_0) \\ &= n \ln \frac{\text{RSS}(s)}{\text{RSS}(s_0)} + (|s| - p_0) \ln n + 2(|s| - p_0) \ln p \\ &\geq n \ln \frac{\text{RSS}(s)}{\text{RSS}(s_0)} - 3p_0 \ln p = n \ln \left(1 + \frac{\text{RSS}(s) - \text{RSS}(s_0)}{\text{RSS}(s_0)}\right) - 3p_0 \ln p \end{aligned}$$

The last inequality follows from $|s| - p_0 \geq -p_0$ and $\ln p > \ln n$. First we estimate $\text{RSS}(s_0)$.

$$\text{RSS}(s_0) = y^T [I - H(s_0)] y = \varepsilon^T [I - H(s_0)] \varepsilon = \varepsilon^T A \varepsilon,$$

where $A = I - H(s_0)$. We have $A^T = A$ and $A^2 = A$, because it is the projection matrix. In addition $\text{Tr } A = j$, where $j = n - p_0$ is a rank of A . To estimate $\varepsilon^T A \varepsilon$, we have to calculate:

$$\begin{aligned} \mathbb{E}(\varepsilon^T A \varepsilon) &= \text{Tr}(A \text{Var} \varepsilon) + (\mathbb{E} \varepsilon)^T A \mathbb{E} \varepsilon = \text{Tr } A = j \\ \text{Var}(\varepsilon^T A \varepsilon) &= [\mathbb{E} \varepsilon^4 - 3(\mathbb{E} \varepsilon^2)^2] \sum_{i=1}^n A_{ii}^2 + [(\mathbb{E} \varepsilon^2)^2 - 1](\text{Tr } A)^2 + 2(\mathbb{E} \varepsilon^2)^2 \text{Tr } A^2 \\ &\leq cj(1 + o_P(1)) \end{aligned}$$

for some constant c , because $\mathbb{E} \varepsilon^4 < \infty$, $\text{Var} \varepsilon = I$ (the covariance matrix), $\mathbb{E} \varepsilon^2 = 1$, $\text{Tr } A^2 = \text{Tr } A = j$ and $\sum_{i=1}^n A_{ii}^2 \leq \sum_{i,j=1}^n A_{ij}^2 = \text{Tr}(A^T A) = \text{Tr } A = j$. Now, using the Chebyshev inequality, we have for arbitrarily small a :

$$\mathbb{P} \left(\left| \frac{\varepsilon^T A \varepsilon}{j} - 1 \right| > a \right) \leq \frac{\text{Var} \frac{\varepsilon^T A \varepsilon}{j}}{a^2} = \frac{cj(1 + o_P(1))}{j^2 a^2}. \quad (5)$$

If n tends to infinity, the probability of (5) converges to zero, thus

$$\varepsilon^T A \varepsilon = j(1 + o_P(1)) = (n - p_0)(1 + o_P(1)) = n(1 + o_P(1)).$$

To estimate $\text{RSS}(s) - \text{RSS}(s_0)$, let us write this difference in the following way:

$$\text{RSS}(s) - \text{RSS}(s_0) = \mu^T [I - H(s)] \mu + 2\mu^T [I - H(s)] \varepsilon + \varepsilon^T H(s_0) \varepsilon - \varepsilon^T H(s) \varepsilon, \quad (6)$$

which is easy to obtain, using the formula $y = \mu + \varepsilon$ and $\mu^T [I - H(s)] \varepsilon = e^T [I - H(s)] \mu$ (it is a number). Let us estimate

$$\max\{\varepsilon^T H(s) \varepsilon : |s| \leq kp_0\}.$$

Using the second part of the lemma for $\mathcal{A} = \{H(s) : |s| \leq r = kp_0\}$, we get

$$\max\{\varepsilon^T H(s) \varepsilon : |s| \leq kp_0\} \leq 2kp_0(\ln p + \sqrt{2 \ln p}) + o_P(1) = O_P(kp_0 \ln p).$$

Now we show that $|\mu^T [I - H(s)] \varepsilon| = \sqrt{\Delta(s) O(kp_0 \ln p)}$. To this aim, note that

$$\mu^T [I - H(s)] \varepsilon = \sqrt{\Delta(s)} \varepsilon'(s),$$

where $\varepsilon'(s)$ is a single subgaussian variable, for which we have $\mathbb{E} \varepsilon'(s) = 0$ and $\text{Var} \varepsilon'(s) = 1$. This is due to the fact that the linear combination of independent subgaussian variables is a subgaussian variable and $\mathbb{E}[\mu^T [I - H(s)] \varepsilon] = 0$ and $\text{Var}[\mu^T [I - H(s)] \varepsilon] = \Delta(s)$. We can write

$$\max\{[\varepsilon'(s)]^2 : |s| \leq kp_0\} \leq \max\{\delta^T I(s) \delta : |s| \leq kp_0\},$$

where $I(s)$ is a n by n matrix, consisting of $|s|$ ones on the diagonal and zeros elsewhere, and δ is a vector of length n containing independent subgaussian

variables with the expected value equal to 0 and the variance equal to 1. For such matrices assumptions of the lemma are satisfied, therefore, using the second part of the lemma, we can write

$$\max\{[\varepsilon'(s)]^2 : |s| \leq kp_0\} = O_P(kp_0 \ln p).$$

Hence,

$$|\mu^T[I - H(s)]\varepsilon| \leq \sqrt{\Delta(s)} \max\{|\varepsilon'(s)| : |s| \leq kp_0\} = \sqrt{\Delta(s)} O(kp_0 \ln p).$$

In conclusion, we get the following estimate:

$$\begin{aligned} & mBIC(s) - mBIC(s_0) \\ & \geq n \ln \left(1 + \frac{\text{RSS}(s) - \text{RSS}(s_0)}{\text{RSS}(s_0)} \right) - 3p_0 \ln p \\ & = n \ln \left(1 + \frac{\Delta(s) + \sqrt{\Delta(s)} O_P(kp_0 \ln p) + p_0(1 + o_P(1)) - O_P(kp_0 \ln p)}{n(1 + o_P(1))} \right) - 3p_0 \ln p \\ & = n \ln \left(1 + \frac{\Delta(s)}{n} (1 + o_P(1)) \right) - 3p_0 \ln p \\ & \geq n \ln \left(1 + \frac{Cp_0 \ln p}{n} (1 + o_P(1)) \right) - 3p_0 \ln p \\ & = Cp_0 \ln p \ln \left(1 + \frac{Cp_0 \ln p}{n} (1 + o_P(1)) \right)^{\frac{n}{Cp_0 \ln p}} - 3p_0 \ln p, \end{aligned}$$

where C is arbitrarily large. The expression $\ln \left(1 + \frac{Cp_0 \ln p}{n} (1 + o_P(1)) \right)^{\frac{n}{Cp_0 \ln p}}$ converges to 1 because $p_0 \ln p = o(n)$, so the above difference is greater than zero if n is big enough.

* * *

Let now $s_0 \subset s$. We have $[I - H(s)]X(s_0) = 0$, therefore

$$\begin{aligned} \text{RSS}(s) &= y^T[I - H(s)]y = \varepsilon^T[I - H(s)]\varepsilon \\ \text{RSS}(s_0) - \text{RSS}(s) &= \varepsilon^T[I - H(s_0)]\varepsilon - \varepsilon^T[I - H(s)]\varepsilon = \varepsilon^T[H(s) - H(s_0)]\varepsilon \end{aligned}$$

We can write

$$\begin{aligned} -n \ln \frac{\text{RSS}(s_0)}{\text{RSS}(s)} &= -n \ln \left(1 + \frac{\text{RSS}(s_0) - \text{RSS}(s)}{\text{RSS}(s_0) - [\text{RSS}(s_0) - \text{RSS}(s)]} \right), \\ &\geq \frac{-n[\text{RSS}(s_0) - \text{RSS}(s)]}{\text{RSS}(s_0) - [\text{RSS}(s_0) - \text{RSS}(s)]} \end{aligned}$$

As before, $\text{RSS}(s_0) = \varepsilon^T[I - H(s_0)]\varepsilon = n(1 + o_P(1))$. We have $\text{RSS}(s_0) - \text{RSS}(s) = \varepsilon^T[H(s) - H(s_0)]\varepsilon$, so we need to estimate $\max\{\varepsilon^T[H(s) - H(s_0)]\varepsilon\}$,

where the maximum is taken over all models of fixed size . Using the lemma for $A_j = H(s) - H(s_0)$, $j = |s| - p_0$ and $r = kp_0 - p_0$, we get

$$\max\{\varepsilon^T[H(s) - H(s_0)]\varepsilon : |s| = j + p_0\} \leq m_j + o_P(1),$$

thus

$$\begin{aligned} -n \ln \frac{\text{RSS}(s_0)}{\text{RSS}(s)} &\geq \frac{-n[\text{RSS}(s_0) - \text{RSS}(s)]}{\text{RSS}(s_0) - [\text{RSS}(s_0) - \text{RSS}(s)]} \\ &\geq \frac{-n(m_j + o_P(1))}{n - m_j + o_P(1)} = \frac{-m_j - o_P(1)}{1 - m_j/n + o_P(1)/n} \geq -m_j + o_P(1). \end{aligned}$$

We can write

$$\begin{aligned} \text{mBIC}(s) - \text{mBIC}(s_0) &\geq -n \ln \frac{\text{RSS}(s_0)}{\text{RSS}(s)} + (|s| - p_0) \ln n + 2(|s| - p_0) \ln p \\ &\geq -m_j + o_P(1) + j \ln n + 2j \ln p \\ &= -2j(\ln p + \sqrt{2 \ln p}) + o_P(1) + j \ln n + 2j \ln p. \end{aligned}$$

The above difference is greater than 0 if n is big enough, because we assumed that $\sqrt{\ln p} = o(\ln n)$. \square

With a little stronger assumptions about p_0 , we can also prove the consistency of mBIC2.

Theorem 2. Assume the model (1), $p > n$, $p = o(n^{\ln n})$, $p_0 \leq \frac{1}{k}\sqrt{n}$ and

$$\lim_{n \rightarrow \infty} \min \left\{ \frac{\Delta(s)}{p_0 \ln p} : s_0 \not\subset s, v(s) \leq kp_0 \right\} = \infty,$$

where k is any constant. Then, for n tends to infinity,

$$\mathbb{P} \left(\min_{\substack{s: v(s) \leq kp_0 \\ s \neq s_0}} \text{mBIC 2}(s) > \text{mBIC 2}(s_0) \right) \rightarrow 1.$$

Proof. If $s_0 \not\subset s$, we can write

$$\begin{aligned} \text{mBIC 2}(s) - \text{mBIC 2}(s_0) &= \text{mBIC}(s) - \text{mBIC}(s_0) + 2 \ln p_0! - 2 \ln |s|! \\ &\geq Cp_0 \ln p \ln \left(1 + \frac{Cp_0 \ln p}{n} (1 + o_P(1)) \right)^{\frac{n}{Cp_0 \ln p}} - (2k + 3)p_0 \ln p, \end{aligned}$$

because $\ln |s|! \leq \ln kp_0! \leq kp_0 \ln kp_0 \leq kp_0 \ln p$. Thus, as in the case of mBIC, the difference $\text{mBIC 2}(s) - \text{mBIC 2}(s_0)$ is greater than 0 if n is big enough (we only need a larger constant C).

If $s_0 \subset s$, we have to estimate the difference $\ln |s|! - \ln p_0!$ more carefully. We have

$$\begin{aligned}\ln |s|! - \ln p_0! &= \ln(j + p_0)! - \ln p_0! = \ln(1 + p_0) + \dots + \ln(j + p_0) \\ &\leq j \ln(j + p_0) \leq j \ln(kp_0)\end{aligned}$$

and then

$$\begin{aligned}\text{mBIC } 2(s) - \text{mBIC } 2(s_0) &= \text{mBIC}(s) - \text{mBIC}(s_0) + 2 \ln p_0! - 2 \ln |s|! \\ &\geq -2j(\ln p + \sqrt{2 \ln p}) + o_P(1) + j \ln n + 2j \ln p - 2j \ln(kp_0).\end{aligned}$$

The above difference is greater than 0 if n is big enough because $\sqrt{\ln p} = o(\ln n)$ and $2j \ln(kp_0) \leq j \ln n$. \square

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